

# A construction of Hopf algebras in braided monoidal category by weak Hopf algebras

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## ABSTRACT

Let  $(H, \mathcal{R})$  be a quasitriangular weak Hopf algebra,  $A$  a weak Hopf algebra, and  $f$  a weak Hopf algebra map between  $H$  and  $A$ . Then we show that  $A$  induce a Hopf algebra  $C_A(A_s)$  in the category  ${}_H\mathcal{M}$ , which generalizes the transmutation theory introduced by Majid. Furthermore, we construct a Hopf algebra  $C_H(H_s)_F$  in the category  ${}_H\mathcal{M}_F$  for any cocommutative weak Hopf algebra  $H$  and a weak invertible unit 2-cocycle  $F$ , which generalizes the result in [5]. Finally, we consider the relation between  $C_H(H_s)_F$  and  $C_{\tilde{H}}(\tilde{H}_s)$ , and obtain that they are isomorphic as objects in the category  $_{\tilde{H}}\mathcal{M}$ , where  $(\tilde{H}, \tilde{\mathcal{R}})$  is a new quasitriangular weak Hopf algebra induced by  $(H, \mathcal{R})$ .

**Key words:** Weak Hopf algebras; Quasitriangular weak Hopf algebras; Weak invertible unit 2-cocycles; The category of left  $H$ -modules.

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## 0. Introduction

Hopf algebras in braided categories, introduced by S. Majid [10, 11], have been extensively studied in the past few years, and have played a very important role in physics and mathematics. S. Majid introduced a method so-called transmutation to construct Hopf algebras in categories. We want to do the similar work in the weak Hopf algebras case. Let  $(H, \mathcal{R})$  be a quasitriangular weak Hopf algebra, and  $A$  a weak Hopf algebra, and there is a weak Hopf algebra map between them. Although we could not obtain  $A$  is a Hopf algebra in the category  ${}_H\mathcal{M}$  of left  $H$ -modules, we prove that the centralizer algebra  $C_A(A_s)$  is a Hopf algebra in  ${}_H\mathcal{M}$ .

On the other hand, Hopf algebras in symmetric monoidal categories came from the deformation-quantization of triangular solutions of the classical Yang-Baxter equations [8]. They called them  $S$ -Hopf algebras, which is obtained by means of an element  $F$  constructed by Drinfel'd in [5]. D. Gurevich and S. Majid [7] pointed out that the method can be used quite generally and define an  $S$ -Hopf algebra  $H_F$  for any pair  $(H, F)$ , where  $H$  is a cocommutative Hopf algebra and  $F$  satisfies a cocycle condition. So, there is a natural question whether this method can be generalized to the weak Hopf algebras setting. In this paper, we obtained a Hopf algebra  $C_H(H_s)_F$  in the category  ${}_H\mathcal{M}_F$ , where  $H$  is a weak Hopf algebra and  $F$  is a weak invertible unit 2-cocycle.

The twisting construction, introduced by Drinfel'd [6], provides a way of obtaining new quasitriangular Hopf algebras from the given ones and the special elements. In 2008, Chen [3] generalized the twisting theory to the weak setting. Applying our theory to the new quasitriangular weak Hopf algebras  $(\tilde{H}, \tilde{\mathcal{R}})$ , we get a Hopf algebra  $C_{\tilde{H}}(\tilde{H}_s)$  in the category  ${}_{\tilde{H}}\mathcal{M}$  of the left  $\tilde{H}$ -modules. The aim and motivation of this paper is to study the relationship between the two Hopf algebras  $C_{\tilde{H}}(\tilde{H}_s)$  and  $C_H(H_s)_F$ , and the categories they contained.

The paper is organized as follows.

In Section 1, we recall some basic notions and results for weak Hopf algebras and braided monoidal categories. In Section 2, we construct a Hopf algebra  $C_A(A_s)$  in the category  ${}_H\mathcal{M}$  of left weak  $H$ -modules, for a quasitriangular weak Hopf algebra  $H$  and a weak Hopf algebra  $A$ . In Section 3, we construct a Hopf algebra  $C_H(H_s)_F$  in the category  ${}_H\mathcal{M}_F$ , for any cocommutative weak Hopf algebra  $H$  and a weak invertible unit 2-cocycle  $F$ . In Section 4, we discuss the relation between  $C_H(H_s)_F$  and  $C_{\tilde{H}}(\tilde{H}_s)$ , where  $(\tilde{H}, \tilde{\mathcal{R}})$  is a new quasitriangular weak Hopf algebra obtained from  $(H, \mathcal{R})$  by the weak twisting theory.

# 1. Preliminaries

Throughout the paper, we let  $k$  be a fixed field, and use the Sweedler formal sum notation for the comultiplication  $\Delta$  over a coalgebra  $C$  with a counit  $\varepsilon_C$  [14], that is,

$$\Delta(c) = c_1 \otimes c_2,$$

for any  $c \in C$ .

## 1.1 Basic definitions and properties about weak Hopf algebras

Recall from Böhm et al. [2] that a weak bialgebra  $H$  is an algebra  $(H, \mu, \eta)$  and a coalgebra  $(H, \Delta, \varepsilon)$  such that  $\Delta(hl) = \Delta(h)\Delta(l)$  and

$$\begin{aligned}\Delta^2(1) &= 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes 1'_1 1_2 \otimes 1'_2 = 1_1 \otimes 1_2 \otimes 1_3, \\ \varepsilon(hgl) &= \varepsilon(hg_1)\varepsilon(g_2l) = \varepsilon(hg_2)\varepsilon(g_1l),\end{aligned}$$

for all  $h, g, l \in H$ .

Moreover, a weak bialgebra  $H$  is called a weak Hopf algebra if there exists a  $k$ -linear map  $S : H \longrightarrow H$ , satisfying the following conditions:

$$S * id = \varepsilon_s, \quad id * S = \varepsilon_t, \quad S * id * S = S,$$

where  $*$  is the usual convolution product, and the idempotent maps  $\varepsilon_t, \varepsilon_s$  are defined by

$$\varepsilon_t(h) = \varepsilon(1_1 h)1_2, \quad \varepsilon_s(h) = 1_1 \varepsilon(h 1_2), \quad \text{for all } h \in H.$$

In this case,  $S$  is called the antipode,  $\varepsilon_t$  is called the target map, and  $\varepsilon_s$  is called the source map, respectively. If antipode  $S$  exists, then it is unique and  $S$  is an anti-algebra and anti-coalgebra morphism. We will always assume that  $S$  is bijective. If  $H$  is a finite dimensional weak Hopf algebra, then  $S$  is automatically bijective.

The target space  $H_t$  and source space  $H_s$  are the images of  $\varepsilon_t$  and  $\varepsilon_s$ , respectively, which can be described as follows:

$$\begin{aligned}H_t &= \{h \in H \mid \varepsilon_t(h) = h\} = \{h \in H \mid \Delta(h) = 1_1 h \otimes 1_2 = h 1_1 \otimes 1_2\}, \\ H_s &= \{h \in H \mid \varepsilon_s(h) = h\} = \{h \in H \mid \Delta(h) = 1_1 \otimes h 1_2 = 1_1 \otimes 1_2 h\}.\end{aligned}$$

It is easy to see that  $H_t$  and  $H_s$  are subalgebras of  $H$ .

Similarly, we have

$$\bar{\varepsilon}_s(h) = \varepsilon(h 1_1)1_2; \quad \bar{\varepsilon}_t(h) = 1_1 \varepsilon(1_2 h).$$

Let  $H$  be a weak Hopf algebra. Recall from Chen [3], we say an element  $F \in \Delta(1)(H \otimes H)\Delta^{cop}(1)$  is a weak invertible unit 2-cocycle if it has the following properties:

- (1) There exists an element  $F^{-1} \in \Delta^{cop}(1)(H \otimes H)\Delta(1)$ , such that

$$FF^{-1} = \Delta(1), \quad F^{-1}F = \Delta^{cop}(1).$$

- (2)  $((\Delta \otimes id)F)F_{12} = ((id \otimes \Delta)F)F_{23}$ , where  $F_{23} = 1 \otimes F$ ,  $F_{12} = F \otimes 1$ .

- (3) For all  $y \in H_s$ , and  $z \in H_t$ , the following equations hold:

$$\begin{aligned} (1 \otimes y)F &= F(y \otimes 1), & (z \otimes 1)F &= F(1 \otimes z); \\ F^{-1}(1 \otimes y) &= (y \otimes 1)F^{-1}, & F^{-1}(z \otimes 1) &= (1 \otimes z)F^{-1}; \\ (1 \otimes y)F^{-1} &= (S^{-1}(y) \otimes 1)F^{-1}, & F(z \otimes 1) &= F(1 \otimes S^{-1}(z)). \end{aligned}$$

We list some equivalent forms of  $((\Delta \otimes id)F)F_{12} = ((id \otimes \Delta)F)F_{23}$  for convenience as follows:

$$\left\{ \begin{array}{l} f^{-(1)} \otimes f^{(1)} f^{-(2)} \otimes f^{(2)} = F^{-(1)} F_1^{(1)} \otimes F_1^{-(2)} F_2^{(1)} \otimes F_2^{-(2)} F^{(2)}, \\ f^{(1)} \otimes f^{(2)} f^{-(1)} \otimes f^{-(2)} = F_1^{-(1)} F^{(1)} \otimes F_2^{-(1)} F_1^{(2)} \otimes F^{-(2)} F_2^{(2)}, \\ F^{-(1)} \otimes f^{-(1)} F_1^{-(2)} \otimes f^{-(2)} F_2^{-(2)} = f^{-(1)} F_1^{-(1)} \otimes f^{-(2)} F_2^{-(1)} \otimes F^{-(2)}, \end{array} \right.$$

where  $f = F$  and  $f^{-1} = F^{-1}$ .

We recall from Nikshych et al. [13] that a quasitriangular weak Hopf algebra is a pair  $(H, \mathcal{R})$ , where  $H$  is a weak Hopf algebra, and  $\mathcal{R} = R^{(1)} \otimes R^{(2)} \in \Delta^{cop}(1)(H \otimes_k H)\Delta(1)$ , satisfying the following conditions:

$$(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \quad (\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23},$$

$$\Delta^{cop}(h)\mathcal{R} = \mathcal{R}\Delta(h), \quad \text{for all } h \in H,$$

where  $\mathcal{R}_{12} = \mathcal{R} \otimes 1$ ,  $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$  etc., as usual, and such that there exists  $\mathcal{R}^{-1} = R^{-(1)} \otimes R^{-(2)} \in \Delta(1)(H \otimes_k H)\Delta^{cop}(1)$  with

$$\mathcal{R}\mathcal{R}^{-1} = \Delta^{cop}(1), \quad \mathcal{R}^{-1}\mathcal{R} = \Delta(1).$$

Next, we summarize some important properties of quasitriangular weak Hopf algebras from [13] as follows, which is needed for our calculation through the paper.

Let  $(H, \mathcal{R})$  be a quasitriangular weak Hopf algebra. Then we have the following equations:

$$\begin{aligned} (1 \otimes z)\mathcal{R} &= \mathcal{R}(z \otimes 1), & (y \otimes 1)\mathcal{R} &= \mathcal{R}(1 \otimes y), \\ (z \otimes 1)\mathcal{R} &= (1 \otimes s(z))\mathcal{R}, & (1 \otimes y)\mathcal{R} &= (s(y) \otimes 1)\mathcal{R}, \\ \mathcal{R}(1 \otimes z) &= \mathcal{R}(S^{-1}(z) \otimes 1), & \mathcal{R}(y \otimes 1) &= \mathcal{R}(1 \otimes S^{-1}(y)), \\ (\varepsilon_s \otimes id)(\mathcal{R}) &= \Delta(1), & (id \otimes \varepsilon_s)(\mathcal{R}) &= (S \otimes id)\Delta^{cop}(1), \\ (\varepsilon_t \otimes id)(\mathcal{R}) &= \Delta^{cop}(1), & (id \otimes \varepsilon_t)(\mathcal{R}) &= (S \otimes id)\Delta(1), \\ (S \otimes id)(\mathcal{R}) &= (id \otimes S^{-1})(\mathcal{R}) = \mathcal{R}^{-1}, & (S \otimes S)(\mathcal{R}) &= \mathcal{R}, \end{aligned}$$

for all  $y \in H_s, z \in H_t$ .

For all  $h \in H$ , we have  $S^2(h) = uhu^{-1}$ , where  $u = S(R^{(2)})R^{(1)}$  is an invertible element of  $H$  such that

$$u^{-1} = R^{(2)}S^2(R^{(1)}), \quad \Delta(u) = \mathcal{R}^{-1}\mathcal{R}_{21}^{-1}(u \otimes u).$$

## 1.2 Some definitions in braided monoidal categories

Let  $(\mathcal{C}, \otimes, I, a, l, r, C)$  denote a braided monoidal category with tensor product  $\otimes$ , base object  $I$  and a braiding  $C$ . Recall that algebras, coalgebras and Hopf algebras in braided monoidal category  $(\mathcal{C}, \otimes, I, C)$ , see [9].

An algebra in  $\mathcal{C}$  is a triple  $A = (A, \eta, \mu)$  where  $A$  is an object in  $\mathcal{C}$  and  $\eta : I \longrightarrow A$  and  $\mu : A \otimes A \longrightarrow A$  are morphisms in  $\mathcal{C}$  such that  $\mu \circ (A \otimes \eta) = id_A = \mu \circ (\eta \otimes A)$ , and  $\mu \circ (A \otimes \mu) = \mu \circ (\mu \otimes A)$ .

A coalgebra in  $\mathcal{C}$  is a triple  $C = (C, \varepsilon, \Delta)$  where  $C$  is an object in  $\mathcal{C}$  and  $\varepsilon : C \longrightarrow I$  and  $\Delta : C \longrightarrow C \otimes C$  are morphisms in  $\mathcal{C}$  such that  $(C \otimes \varepsilon) \circ \Delta = id_C = (\varepsilon \otimes C) \circ \Delta$ , and  $(C \otimes \Delta) \circ \Delta = (\Delta \otimes C) \circ \Delta$ .

We call  $(H, \mu, \eta, \Delta, \varepsilon, S)$  is a bialgebra in a braided monoidal category  $(\mathcal{C}, \otimes, I, C)$  if  $(H, \mu, \eta)$  is an algebra in  $\mathcal{C}$ ,  $(H, \Delta, \varepsilon)$  is a coalgebra in  $\mathcal{C}$  and also satisfies

$$\begin{aligned} \Delta \circ \mu &= (\mu \otimes \mu) \circ (H \otimes C \otimes H) \circ (\Delta \otimes \Delta), \\ \varepsilon \circ \mu &= \varepsilon \otimes \varepsilon, \quad \Delta(1) = 1 \otimes 1. \end{aligned}$$

A bialgebra  $(H, \mu, \eta, \Delta, \varepsilon)$  in  $\mathcal{C}$  is called a Hopf algebra in  $\mathcal{C}$  if there exists a morphism  $S : H \longrightarrow H$  in  $\mathcal{C}$  satisfying

$$\mu \circ (S \otimes H) \circ \Delta = \varepsilon \circ \eta = \mu \circ (H \otimes S) \circ \Delta.$$

This definition is due to Majid [12].

## 2. Constructing Hopf algebras in braided monoidal category by quasitriangular structures

Let  $(H, \mu, \eta, \Delta, \varepsilon, S, \mathcal{R})$  be a quasitriangular weak Hopf algebra and  ${}_H\mathcal{M}$  denote the category of left  $H$ -modules. Then we have a monoidal category  $({}_H\mathcal{M}, \otimes_t, H_t, a, l, r)$  [13], where  $l$  and  $r$  are defined as follows, for any  $V \in {}_H\mathcal{M}$ ,  $v \in V$ ,  $z \in H_t$ ,

$$l_V(z \otimes_t v) = z \cdot v, \quad r_v(v \otimes_t z) = S^{-1}(z) \cdot v.$$

Furthermore, the category  $({}_H\mathcal{M}, \otimes_t, H_t, a, l, r)$  is braided, the braiding is defined as follows:

$$\Psi_{V,W} : V \otimes_t W \longrightarrow W \otimes_t V, \quad v \otimes_t w \longmapsto R^{(2)} \cdot w \otimes R^{(1)} \cdot v.$$

Let  $(A, \underline{\mu}, \underline{\eta}, \underline{\Delta}, \underline{\varepsilon}, \underline{S})$  be a weak Hopf algebra and  $f$  a weak Hopf algebra map between  $H$  and  $A$ . Then we consider the centralizer of  $A_s$  in  $A$ , denoted by

$$B = C_A(A_s) = \{b \in A \mid bx = xb, \text{ for all } x \in A_s\},$$

Then we have the following proposition.

**Proposition 2.1.**  $B$  is an object in  ${}_H\mathcal{M}$ , where the left  $H$ -module structure on  $B$  is defined by  $h \cdot b = f(h_1)bf(S(h_2))$ , for all  $h \in H$ ,  $b \in B$ . Moreover,  $B$  also is a left weak  $H$ -module algebra.

**Proof.** We check first that  $B$  satisfies the module conditions. For all  $h, g \in H$ , and  $b \in B$ , we have

$$\begin{aligned} (gh) \cdot b &= f(g_1)f(h_1)bf(S(h_2))f(S(g_2)) \\ &= f(g_1)(h \cdot b)f(S(g_2)) = g \cdot (h \cdot b), \end{aligned}$$

$$\text{and } 1 \cdot b = f(1_1)bf(S(1_2)) = bf(1)_1 \underline{S}(f(1)_2) = b.$$

Next we prove that  $B$  is a left weak  $H$ -module algebra. In fact, for all  $h \in H$ ,  $a, b \in B$ ,

$$\begin{aligned} (h_1 \cdot a)(h_2 \cdot b) &= f(h_1)af(S(h_2))f(h_3)bf(S(h_4)) \\ &= f(h_1)af(\varepsilon_s(h_2))bf(S(h_3)) = f(h_1)af(1_1)bf(S(h_21_2)) \\ &= f(h_1)abf(S(h_2)) = h \cdot (ab) \end{aligned}$$

and

$$\begin{aligned} \varepsilon_t(h) \cdot 1_A &= f(h_1S(h_3))f(S(\varepsilon_t(h_2))) = f(h_1S(1_2h_2))f(S(S(1_1))) \\ &= f(h_1)f(S(h_2))f(1_1S(1_2)) = h \cdot 1_A \end{aligned}$$

This finishes the proof of the proposition.

**Proposition 2.2.**  $B$  is an algebra in  ${}_H\mathcal{M}$ . The multiplication and unit are given by:

$$\mu_B : B \otimes_t B \longrightarrow B, \quad a \otimes_t b \longmapsto ab, \quad \text{for all } a, b \in B,$$

and

$$\eta_B : H_t \longrightarrow B, \quad x \longmapsto f(x), \quad \text{for all } x \in H_t,$$

respectively.

**Proof.** We first prove that the definition is well-defined. Obviously,  $ab \in B$ ,  $\mu_B(a \otimes_t b) = (1_1 \cdot b)(1_2 \cdot b) = 1 \cdot (ab) = ab$  and

$$f(x) = f(\varepsilon_t(x)) = f(1_2)\underline{\varepsilon}f(1_1x) = f(1_2)\underline{\varepsilon}(f(1_1)f(x)) = \underline{\varepsilon}_t(f(x)),$$

Thus we have  $f(x) \in A_t \subset B$ . It is easy to see that the associativity are straightforward and the multiplication and unit are morphisms in  ${}_H\mathcal{M}$ . So it remains to verify the unity. In fact, for all  $z \in H_t, a \in B$ , we have

$$\mu_B \circ (\eta_B \otimes id)(z \otimes_t a) = f(z)a = f(1_1z)af(s(1_2)) = z \cdot a,$$

and

$$\mu_B \circ (id \otimes \eta_B)(a \otimes_t z) = af(z) = a \cdot z.$$

This completes the proof.

**Proposition 2.3.**  $B$  is a coalgebra in  ${}_H\mathcal{M}$ . The comultiplication and counit are given by

$$\Delta_B : B \longrightarrow B \otimes_t B, \quad b \longmapsto b_1f(S(R^{(2)})) \otimes R^{(1)} \cdot b_2,$$

and

$$\varepsilon_B : B \longrightarrow H_t, \quad b \longmapsto \underline{\varepsilon}(f(1_1)b)1_2, \quad \text{for all } b \in B,$$

respectively.

**Proof:** Firstly, we verify that  $\Delta_B$  and  $\varepsilon_B$  are well-defined. Obviously,  $\varepsilon_B(b) \in H_t$ . For all  $b \in B$ , we have

$$\begin{aligned} \Delta_B(b) &= b_1f(S(R^{(2)})) \otimes R^{(1)} \cdot b_2 \\ &= b_1f(S(r^{(2)}))f(S(R^{(2)})) \otimes f(R^{(1)})b_2f(S(r^{(1)})) \\ &= f(1_1)b_1f(S(r^{(2)}))f(S(R^{(2)})) \otimes f(R^{(1)})f(1_2)b_2f(S(r^{(1)})) \\ &= f(1_1)b_1f(S(r^{(2)}))f(S(R^{(2)})) \otimes f(R^{(1)}1_2)b_2f(S(r^{(1)})) \\ &= f(1_1)b_1f(S(r^{(2)}))f(S(1_2R^{(2)})) \otimes f(R^{(1)})b_2f(S(r^{(1)})) \\ &= f(1_1)b_1f(S(R^{(2)}))f(S(1_2)) \otimes R^{(1)} \cdot b_2 \\ &= f(1)_1b_1f(S(R^{(2)}))\underline{S}(f(1)_2) \otimes f(1'_1)(R^{(1)} \cdot b_2)f(S(1'_2)) \\ &\in B \otimes B. \end{aligned}$$

and

$$\begin{aligned} \Delta_B(b) &= b_1f(S(R^{(2)})) \otimes f(R^{(1)}_1)b_2f(S(R^{(1)}_2)) \\ &= f(1_1)b_1f(S(R^{(2)}r^{(2)})) \otimes f(R^{(1)})f(1_2)b_2f(S(r^{(1)})) \\ &= f(1_1)b_1f(S(R^{(2)}r^{(2)})) \otimes f(R^{(1)})f(S^{-1}\varepsilon_s(1_2))b_2f(S(r^{(1)})) \\ &= f(1_1)b_1f(S(R^{(2)}r^{(2)})) \otimes f(R^{(1)})f(S^{-1}(1_3)1_2)b_2f(S(r^{(1)})) \\ &= f(1_1)b_1f(S(R^{(2)}1_3r^{(2)})) \otimes f(R^{(1)})f(1_2)b_2f(S(r^{(1)})) \end{aligned}$$

$$\begin{aligned}
&= f(1_1)b_1f(S(R^{(2)}1_3S(1_4)r^{(2)})) \otimes f(R^{(1)}1_2)b_2f(S(r^{(1)})) \\
&= f(1_1)b_1f(S(1_2R^{(2)}S(1_4)r^{(2)})) \otimes f(1_3R^{(1)})b_2f(S(r^{(1)})) \\
&= f(1_1)b_1f(S(1_2R^{(2)}r^{(2)})) \otimes f(1_3)f(R^{(1)})b_2f(S(1_4r^{(1)})) \\
&= f(1_1)b_1f(S(R^{(2)}))f(S(1_2)) \otimes f(1_3)f(R^{(1)})b_2f(S(R^{(1)}))f(S(1_4)) \\
&= 1_1 \cdot (b_1f(S(R^{(2)})) \otimes 1_2 \cdot (R^{(1)} \cdot b_2) \\
&= b_1f(S(R^{(2)})) \otimes_t R^{(1)} \cdot b_2 \\
&\in B \otimes_t B
\end{aligned}$$

Secondly, the coassociativity is same as the setting of Hopf algebras, then we check the counity as follows:

$$\begin{aligned}
&(\varepsilon_B \otimes id_B) \circ \Delta_B(b) \\
&= \varepsilon_B(b_1f(S(r^{(2)}))f(S(R^{(2)}))) \otimes_t f(R^{(1)})b_2f(S(r^{(1)})) \\
&= \underline{\varepsilon}(f(1_1)b_1f(S(r^{(2)}))f(S(R^{(2)})))1_2 \cdot [f(R^{(1)})b_2f(S(r^{(1)}))] \\
&= \underline{\varepsilon}(f(1_1)b_1f(S(r^{(2)}))f(S(R^{(2)}))f(1_2)f(R^{(1)})b_2f(S(1_3r^{(1)}))) \\
&= \underline{\varepsilon}(f(1_1)b_1f(S(S(1_3)r^{(2)}))f(S(R^{(2)}))f(1_2)f(R^{(1)})b_2f(S(r^{(1)}))) \\
&= \underline{\varepsilon}(f(1_1)b_1f(S(r^{(2)}))f(S^2(1_3R^{(2)}))f(1_2S(R^{(1)}))b_2f(S(r^{(1)}))) \\
&= \underline{\varepsilon}(f(1_1)b_1f(S(r^{(2)}))f(S^2(R^{(2)}))f(1_2S(R^{(1)}1_3))b_2f(S(r^{(1)}))) \\
&= \underline{\varepsilon}(f(1_1)b_1f(S(r^{(2)}))f(S(R^{(2)}))f(\varepsilon_t(1_2))f(R^{(1)})b_2f(S(r^{(1)}))) \\
&= \underline{\varepsilon}(f(1_1)b_1f(S(r^{(2)}))f(S(R^{(2)}))f(1_2)f(R^{(1)})b_2f(S(r^{(1)}))) \\
&= \underline{\varepsilon}_t(b_1f(S(r^{(2)}))f(S(R^{(2)})))f(R^{(1)})b_2f(S(r^{(1)})) \\
&= b_1\underline{\varepsilon}_t(f(S(R^{(2)})))\underline{S}(b_2)f(R^{(1)})b_3f(S(R^{(1)})) \\
&= b_1\underline{\varepsilon}(f(1_1S(R^{(2)})))f(1_2)\underline{S}(b_2)f(R^{(1)})b_3f(S(R^{(1)})) \\
&= b_1\varepsilon(1_1S(R^{(2)}))f(1_2)\underline{S}(b_2)f(R^{(1)})b_3f(S(R^{(1)})) \\
&= b_1\varepsilon S(R^{(2)}1_2)f(S(1_1))\underline{S}(b_2)f(R^{(1)})b_3f(S(R^{(1)})) \\
&= b_1\varepsilon(R^{(2)})f(S(1_1))\underline{S}(b_2)f([R^{(1)}S^{-1}(1_2)]_1)b_3f(S([R^{(1)}S^{-1}(1_2)]_2)) \\
&= b_1f(S^2(1_3))\underline{S}(b_2)f(1_1)b_3f(S(1_2)) = b_1\underline{S}(1_{A_1})\underline{\varepsilon}_s(b_2)\underline{\varepsilon}_s(\underline{S}(b_3))1_{A_2} \\
&= b_2\underline{\varepsilon}_s(b_3)\underline{\varepsilon}_s(\underline{S}(b_4))S^{-1}(\underline{\varepsilon}_s(b_1)) = b_2\varepsilon_s(\varepsilon_t(b_3))\overline{\varepsilon}_s(b_1) \\
&= 1_{A_1}b_2\underline{S}(1_{A_2})1'_{A_2}\underline{\varepsilon}(b_11'_{A_1}) = S^{-1}(1_{A_2})b_21_{A_1}1'_{A_2}\underline{\varepsilon}(b_11'_{A_1}) \\
&= S^{-1}(1_{A_2})b1_{A_1} = 1_{A_1}b\underline{S}(1_{A_2}) = b.
\end{aligned}$$

Likewise, on the other side we have

$$\begin{aligned}
&(id_B \otimes \varepsilon_B) \circ \Delta_B(b) \\
&= b_1f(S(r^{(2)}))f(S(R^{(2)})) \otimes_t \underline{\varepsilon}(f(1_1)f(R^{(1)})b_2f(S(r^{(1)})))1_2 \\
&= b_1f(S(r^{(2)}))f(S(R^{(2)}))f(1_2)\underline{\varepsilon}(f(1_1)f(R^{(1)})b_2f(S(r^{(1)}))) \\
&= b_1f(S(r^{(2)}))f(S(R^{(2)}))\underline{\varepsilon}_t(f(R^{(1)})b_2f(S(r^{(1)}))) \\
&= b_1f(S(r^{(2)}))f(S(R^{(2)}))f(R^{(1)})b_2\underline{\varepsilon}_t(f(S(r^{(1)})))\underline{S}(b_3)\underline{S}(f(R^{(1)}))
\end{aligned}$$



$$\begin{aligned}
&= b_1 f(S(r^{(2)})) f(S(x^{(2)})) f(S(R^{(2)})) f(R^{(1)}) b_2 \underline{\varepsilon}_t(f(S(r^{(1)}))) \underline{S}(b_3) \underline{S}(f(x^{(1)})) \\
&= b_1 f(r^{(2)}) f(S(x^{(2)})) f(S(R^{(2)})) f(R^{(1)}) b_2 \underline{\varepsilon}(f(1_1 r^{(1)})) f(1_2) \underline{S}(b_3) \underline{S}(f(x^{(1)})) \\
&= b_1 f(r^{(2)} 1_1) f(S(x^{(2)})) f(S(R^{(2)})) f(R^{(1)}) b_2 \varepsilon(r^{(1)}) f(1_2) \underline{S}(b_3) \underline{S}(f(x^{(1)})) \\
&= b_1 f(1_1) f(S(x^{(2)})) f(S(R^{(2)})) f(R^{(1)}) b_2 f(1_2) \underline{S}(b_3) \underline{S}(f(x^{(1)})) \\
&= b_1 f(S(r^{(2)})) f(S(R^{(2)})) f(R^{(1)}) \underline{\varepsilon}_t(b_2) f(S(r^{(1)})) \\
&= f(1_1) b f(r^{(2)}) f(S(R^{(2)})) f(R^{(1)}) f(1_2 r^{(1)}) \\
&= f(1_1) b f(S(1_2) r^{(2)}) f(S(R^{(2)})) f(R^{(1)}) f(r^{(1)}) \\
&= b f(S(1_1)) f(1_2) = b.
\end{aligned}$$

Thus we obtain  $(\varepsilon_B \otimes id_B) \circ \Delta_B = id_B = (id_B \otimes \varepsilon_B) \circ \Delta_B$ .

Thirdly, we need to prove that  $\Delta_B$  and  $\varepsilon_B$  are morphisms in  $\mathcal{C}$ . For all  $b \in B$ , we do the following calculation:

$$\begin{aligned}
\varepsilon_B(h \cdot b) &= \underline{\varepsilon}(f(1_1) f(h_1) b f(S(h_2))) 1_2 = \underline{\varepsilon}(f(1_1) f(h_1) b \underline{S}(f(h_2))) 1_2 \\
&= \underline{\varepsilon}[f(1_1) f(h_1) b \underline{\varepsilon}_s(f(h_2))] 1_2 = \underline{\varepsilon}[f(1_1) f(h_1) \underline{\varepsilon}_s(f(h_2)) b] 1_2 \\
&= \underline{\varepsilon}(f(1_1 h) b) 1_2 = \underline{\varepsilon}(f(h_1) b) \varepsilon_t(h_2) = \underline{\varepsilon}(f(h_1 1_1)) \underline{\varepsilon}(f(1_2) b) \varepsilon_t(h_2) \\
&= \varepsilon(\varepsilon_s(h_1) 1_1) \underline{\varepsilon}(f(1_2) b) \varepsilon_t(h_2) = \underline{\varepsilon} f(\varepsilon_s(h_1) 1_1) \underline{\varepsilon}(f(1_2) b) \varepsilon_t(h_2) \\
&= \underline{\varepsilon}(f(\varepsilon_s(h_1)) b) \varepsilon_t(h_2) = \underline{\varepsilon}(f(1_1) b) \varepsilon_t(h 1_2) \\
&= \underline{\varepsilon}(f(1_1) b) h_1 \varepsilon_t(1_2) S(h_2) = \underline{\varepsilon}(f(1_1) b) (h \cdot 1_2) \\
&= h \cdot \varepsilon_B(b),
\end{aligned}$$

and  $\Delta_B(h \cdot b) = h \cdot \Delta_B(b)$  is clear. This completes the verification of the coalgebra structure on  $B$ .

**Theorem 2.4.**  $B$  is a bialgebra in  ${}_H\mathcal{M}$  with structures defined in Proposition 2.2 and Proposition 2.3, and we denote  $\Delta_B(b) = b_{\underline{1}} \otimes_t b_{\underline{2}}$ .

**Proof.** We now to check that these structure maps obey the axioms of a bialgebra in  ${}_H\mathcal{M}$ . First, we claim that  $(\mu_B \otimes \mu_B) \circ (id_B \otimes \Psi_{B,B} \otimes id_B) \circ (\Delta_B \otimes \Delta_B) = \Delta_B \circ \mu_B$ , in fact, for all  $a, b \in B$ ,

$$\begin{aligned}
&(\mu_B \otimes \mu_B) \circ (id_B \otimes \Psi_{B,B} \otimes id_B) \circ (\Delta_B \otimes \Delta_B)(a \otimes_t b) \\
&= (\mu_B \otimes \mu_B) \circ (id_B \otimes \Psi_{B,B} \otimes id_B)(a_{\underline{1}} \otimes_t a_{\underline{2}} \otimes_t b_{\underline{1}} \otimes_t b_{\underline{2}}) \\
&= a_1 f(S(r^{(2)} x^{(2)})) b_{\underline{1}} f(S(R^{(2)})) \otimes_t (R^{(1)} \cdot (S(r^{(1)}) x^{(1)} \cdot a_1)) b_{\underline{2}} \\
&= a_1 f(S(1_2)) b_{\underline{1}} f(S(R^{(2)})) \otimes_t (R^{(1)} 1_1 \cdot a_1) b_{\underline{2}} \\
&= a_1 f(S(1_2)) b_{\underline{1}} f(S(R^{(2)} S(1_1))) \otimes_t (R^{(1)} \cdot a_1) b_{\underline{2}} \\
&= a_1 f(1_1) b_{\underline{1}} f(S(1_2)) f(S(R^{(2)})) \otimes_t (R^{(1)} \cdot a_1) b_{\underline{2}} \\
&= a_1 b_{\underline{1}} f(S(R^{(2)})) \otimes_t (R^{(1)} \cdot a_1) b_{\underline{2}} \\
&= a_1 b_1 f(S(r^{(2)})) f(S(R^{(2)})) \otimes_t (R^{(1)} \cdot a_1) (r^{(1)} \cdot b_2) \\
&= a_1 b_1 f(S(R^{(2)})) \otimes_t (R_1^{(1)} \cdot a_1) (R_2^{(1)} \cdot b_2)
\end{aligned}$$

$$\begin{aligned}
&= a_1 b_1 f(S(R^{(2)})) \otimes_t R^{(1)} \cdot a_2 b_2 \\
&= \Delta_B \circ \mu_B(a \otimes_t b).
\end{aligned}$$

Then we prove  $\varepsilon_B \circ \mu_B = \varepsilon_B \otimes \varepsilon_B$  as follows:

$$\begin{aligned}
&\mu_B(\varepsilon_B(a) \otimes_t \varepsilon_B(b)) = \varepsilon_B(a) \varepsilon_B(b) = \underline{\varepsilon}(f(1_1)a)1_2 \underline{\varepsilon}(f(1'_1)b)1'_2 \\
&= \underline{\varepsilon}(f(1_1)a) \underline{\varepsilon}(f(\varepsilon_s(1_2)b)1_3) = \underline{\varepsilon}(f(1_1)a) \underline{\varepsilon}(f(\varepsilon_s(1_2)f(1'_2)) \underline{\varepsilon}(f(1'_1)b)1_3 \\
&= \underline{\varepsilon}(f(1_1)a) \varepsilon(1_2 1'_2) \underline{\varepsilon}(f(1'_1)b)1_3 = \underline{\varepsilon}(f(1_1)a) \underline{\varepsilon}(f(1_2)f(1'_2)) \underline{\varepsilon}(f(1'_1)b)1_3 \\
&= \underline{\varepsilon}(af(1_1)) \underline{\varepsilon}(f(1_2)b)1_3 = \underline{\varepsilon}(af(1_1)b)1_2 = \underline{\varepsilon}(f(1_1)ab)1_2 \\
&= \varepsilon_B(ab) = \varepsilon_B \circ \mu_B(a \otimes_t b).
\end{aligned}$$

Finally, for  $1_A \in B$ , we have

$$\begin{aligned}
\Delta_B(1_A) &= 1_{A_1} f(S(r^{(2)})) f(S(R^{(2)})) \otimes_t f(R^{(1)}) 1_{A_2} f(S(r^{(1)})) \\
&= f(1_1) f(r^{(2)}) f(S(R^{(2)})) \otimes_t f(R^{(1)}) f(1_2 r^{(1)}) \\
&= f(1_1) f(S(1_2) r^{(2)}) f(S(R^{(2)})) \otimes_t f(R^{(1)}) f(r^{(1)}) \\
&= f(S(R^{(2)} r^{(2)})) \otimes_t f(R^{(1)} S(r^{(1)})) \\
&= f(S(1_1)) \otimes_t f(1_2) = f(S(1_1))(1_2 \cdot 1_A) \otimes_t 1_A \\
&= f(S(1_1)) f(1_2) f(S(1_3)) \otimes_t 1_A \\
&= 1_A \otimes_t 1_A.
\end{aligned}$$

This completes our proof.

**Theorem 2.5.** In the situation of Theorem 2.4,  $B$  is a Hopf algebra in  ${}_H\mathcal{M}$  with antipode  $S_B$ , which is given by

$$S_B : B \longrightarrow B, \quad b \longmapsto f(R^{(2)}) \underline{S}(R^{(1)} \cdot b),$$

where the  $\underline{S}$  is the antipode of  $A$ .

**Proof.** For convenience we write  $S_B$  in the following equivalent form

$$S_B(b) = f(u^{-1}) f(S(R^{(2)})) \underline{S}(b) f(R^{(2)}),$$

where  $u^{-1} = \mathcal{R}^{(2)} S^2(\mathcal{R}^{(1)})$  is the inverse of the element  $u = S(\mathcal{R}^{(2)}) \mathcal{R}^{(1)}$  follows Proposition 1.1. First, we assert that  $S_B$  is well-defined. In fact, for all  $b \in B$ , we have

$$\begin{aligned}
&f(1_1) f(u^{-1}) f(S(R^{(2)})) S(b) f(R^{(1)}) f(S(1_2)) \\
&= f(u^{-1}) f(S^2(1_1)) f(S(R^{(2)})) S(b) f(R^{(1)}) f(S(1_2)) \\
&= f(u^{-1}) f(S(R^{(2)} 1_2)) S(b) f(R^{(1)}) f(1_1) \\
&= f(u^{-1}) f(S(R^{(2)})) S(b) f(R^{(1)} S^{-1}(1_2)) f(1_1) \\
&= f(u^{-1}) f(S(R^{(2)})) S(b) f(R^{(1)}).
\end{aligned}$$

Thus,  $S_B(b) \in B$ . It is easy to get  $S_B$  is a morphism in  ${}_H\mathcal{M}$ . Now we prove the properties of antipode. Then for all  $b \in B$ ,

$$\begin{aligned}
& \mu_B \circ (id_B \otimes S_B) \circ \Delta_B(b) \\
&= b_1 f(S(R^{(2)})) f(u^{-1}) f(S(r^{(2)})) \underline{S}(R^{(1)} \cdot b_2) f(r^{(1)}) \\
&= b_1 f(S(R^{(2)})) f(u^{-1} S^2(r^{(2)})) \underline{S}(R^{(1)} \cdot b_2) f(S(r^{(1)})) \\
&= b_1 f(S(R^{(2)})) f(r^{(2)} u^{-1}) \underline{S}(R^{(1)} \cdot b_2) f(S(r^{(1)})) \\
&= b_1 f(S(r^{(2)} R^{(2)})) f(u^{-1}) \underline{S}(R^{(1)} \cdot b_2) f(S^2(r^{(1)})) \\
&= b_1 f(S(R^{(2)})) f(u^{-1}) \underline{S}(f(R_2^{(1)}) b_2 f(S(R_3^{(1)}))) f(S^2(R_1^{(1)})) \\
&= b_1 f(S(R^{(2)})) f(u^{-1}) f(S^2(R_3^{(1)})) \underline{S}(b_2) f(S(R_2^{(1)})) f(S^2(R_1^{(1)})) \\
&= b_1 f(S(R^{(2)})) f(u^{-1}) f(S^2(R_2^{(1)})) \underline{S}(b_2) f(S(\varepsilon_s(R_1^{(1)}))) \\
&= b_1 f(S(R^{(2)})) f(u^{-1}) f(S^2(R^{(1)} 1_2)) \underline{S}(b_2) f(S(1_1)) \\
&= b_1 f(R^{(2)}) f(u^{-1}) f(S(R^{(1)})) f(S^2(1_2)) \underline{S}(b_2) f(S(1_1)) \\
&= b_1 f(R^{(2)}) f(u^{-1}) f(S(R^{(1)})) \underline{S}(f(S(1_2))) \underline{S}(b_2) \underline{S}(f(1_1)) \\
&= b_1 f(u^{-1}) f(S^2(R^{(2)})) f(S(R^{(1)})) \underline{S}(b_2) \\
&= \underline{\varepsilon}_t(b) = \underline{\varepsilon}(f(1_1) b) f(1_2) = \underline{\varepsilon}(f(1_1) b) \eta_B(1_2) \\
&= \eta_B \circ \varepsilon_B(b),
\end{aligned}$$

and

$$\begin{aligned}
& \mu_B \circ (S_B \otimes id_B) \circ \Delta_B(b) = f(u^{-1}) f(S(R^{(2)})) \underline{S}(b_1) b_2 f(R^{(1)}) \\
&= f(u^{-1}) f(S(R^{(2)})) \underline{\varepsilon}(b f(1_2)) f(1_1) f(R^{(1)}) = (u^{-1}) f(S(R^{(2)} 1_1)) f(R^{(1)}) \underline{\varepsilon}(b f(1_2)) \\
&= f(u^{-1}) f(S^2(1_2)) f(u) \underline{\varepsilon}(b f(S(1_1))) = f(1_2) f(u^{-1}) f(u) \underline{\varepsilon}(b f(S(1_1))) \\
&= f(1_2) \underline{\varepsilon}(b \underline{S}(f(1_1))) = f(1_2) \underline{\varepsilon}(b \underline{\varepsilon}_t \circ \underline{\varepsilon}_s(f(1_1))) = \underline{\varepsilon}(f(1_1) b) f(1_2) \\
&= \eta_B \circ \varepsilon_B(b).
\end{aligned}$$

This completes the proof of the theorem.

**Corollary 2.6.** Let  $(H, \mathcal{R})$  be a quasitriangular weak Hopf algebra, then  $C_H(H_s)$  is a Hopf algebra in the category  ${}_H\mathcal{M}$ , where the left  $H$ -module structure on  $C_H(H_s)$  is given by

$$h \cdot g = h_1 g S(h_2), \quad \text{for all } h \in H, g \in C_H(H_s),$$

and we denote this action of  $H$  on  $C_H(H_s)$  by  $Ad$ , i.e.,  $Ad_h(g) = h_1 g S(h_2)$ . The algebra structure is defined as follows:

$$\widehat{\mu}(g \otimes_t l) = gl, \quad \widehat{\eta}(x) = x, \quad \text{for all } g, l \in C_H(H_s), x \in H_t.$$

The comultiplication and counit are defined by

$$\widehat{\Delta}(g) = g_1 S(R^{(2)}) \otimes_t Ad_{R^{(1)}}(g_2) \quad \text{and} \quad \widehat{\varepsilon}(g) = \varepsilon_t(g),$$

respectively.

**Corollary 2.7.** Let  $(H, \mathcal{R})$  be a quasitriangular Hopf algebra,  $A$  a Hopf algebra and  $f$  a Hopf algebra map between them. Then  $C_A(A_s) = A$  and theorem 2.5 is the transmutation theorem in the sense of Majid [10].

### 3. Constructing Hopf algebras in braided monoidal category by weak invertible unit 2-cocycles

In this section, we assume that  $H$  is a cocommutative weak Hopf algebra,  $F = F^{(1)} \otimes F^{(2)} \in \Delta(1)(H \otimes H)\Delta^{cop}(1)$  is a weak invertible unit 2-cocycle, and let  ${}_H\mathcal{M}_F$  denote the category of left  $H$ -modules, which module action is  $Ad$ , braiding and module structure on tensor product are given by

$$\Phi(m \otimes_t n) = Ad_{F^{-(1)}F^{(2)}}(n) \otimes_t Ad_{F^{-(2)}F^{(1)}}(m),$$

and

$$Ad_h(m \otimes_t n) = Ad_{F^{-(1)}h_1F^{(1)}}(m) \otimes_t Ad_{F^{-(2)}h_2F^{(2)}}(n),$$

respectively, for all  $m \in M$ ,  $n \in N$ ,  $M, N \in {}_H\mathcal{M}_F$ . Then we construct a Hopf algebra in the category  ${}_H\mathcal{M}_F$ .

**Proposition 3.1.** Let  $H$  be a cocommutative weak Hopf algebra,  $F$  a weak invertible unit 2-cocycle. Then  $C_H(H_s)$  is an algebra in  ${}_H\mathcal{M}_F$ , which multiplication and unit are defined by

$$\cdot_F : C_H(H_s) \otimes_t C_H(H_s) \longrightarrow C_H(H_s), \quad a \cdot_F b = Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(b),$$

and

$$\eta_F : H_t \longrightarrow C_H(H_s), \quad \eta_F(x) = x,$$

for all  $a, b \in C_H(H_s)$ ,  $x \in H_t$ .

**Proof.** First, we show that the multiplication is well-defined. For all  $a, b \in C_H(H_s)$ ,  $x \in H_s$ , we have

$$\begin{aligned} \cdot_F(a \otimes_t b) &= \cdot_F(Ad_{1_1}(a) \otimes Ad_{1_2}(b)) = Ad_{F^{(1)}}(Ad_{1_1}(a))Ad_{F^{(2)}}(Ad_{1_2}(b)) \\ &= Ad_{F^{(1)}1_1}(a)Ad_{F^{(2)}1_2}(b) = Ad_{F^{(1)}1_2}(a)Ad_{F^{(2)}1_2}(b) = Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(b), \end{aligned}$$

and by

$$\begin{aligned} xAd_{F^{(1)}}(a) &= xF_1^{(1)}aS(F_2^{(1)}) = x1_1F_1^{(1)}aS(F_2^{(1)})S(1_2) \\ &= 1_11_1F_1^{(1)}aS(F_2^{(1)})S(1_2)x = F_1^{(1)}aS(F_2^{(1)})x = Ad_{F^{(1)}}(a)x, \end{aligned}$$

we have  $Ad_{F^{(1)}}(a) \in C_H(H_s)$ . Similarly,  $Ad_{F^{(2)}}(b) \in C_H(H_s)$ , so  $a \cdot_F b \in C_H(H_s)$ .

Secondly, we check the associativity and unity. For all  $a, b, c \in C_H(H_s)$ , we do the following calculation.

$$\begin{aligned}
(a \cdot_F b) \cdot_F c &= Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(b) \cdot_F c = Ad_{f^{(1)}}(Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(b))Ad_{f^{(2)}}(c) \\
&= f_1^{(1)}Ad_{F^{(1)}}(a)1_1F_1^{(2)}bS(F_2^{(2)})S(1_2)S(f_2^{(1)})Ad_{f^{(2)}}(c) \\
&= f_1^{(1)}Ad_{F^{(1)}}(a)1_1Ad_{F^{(2)}}(b)S(f_2^{(1)}1_2)Ad_{f^{(2)}}(c) \\
&= f_1^{(1)}Ad_{F^{(1)}}(a)\varepsilon_s(f_2^{(1)})Ad_{F^{(2)}}(b)S(f_3^{(1)})Ad_{f^{(2)}}(c) \\
&= f_1^{(1)}Ad_{F^{(1)}}(a)S(f_2^{(1)})f_3^{(1)}Ad_{F^{(2)}}(b)S(f_4^{(1)})Ad_{f^{(2)}}(c) \\
&= Ad_{f_1^{(1)}F^{(1)}}(a)Ad_{f_2^{(1)}F^{(2)}}(b)Ad_{f^{(2)}}(c) \\
&= Ad_{f^{(1)}}(a)f_1^{(2)}Ad_{F^{(1)}}(b)S(f_2^{(2)})f_3^{(2)}Ad_{F^{(2)}}(c)S(f_4^{(2)}) \\
&= Ad_{f^{(1)}}(a)f_1^{(2)}Ad_{F^{(1)}}(b)\varepsilon_s(f_2^{(2)})Ad_{F^{(2)}}(c)S(f_3^{(2)}) \\
&= Ad_{f^{(1)}}(a)f_1^{(2)}\varepsilon_s(f_2^{(2)})Ad_{F^{(1)}}(b)Ad_{F^{(2)}}(c)S(f_3^{(2)}) \\
&= Ad_{f^{(1)}}(a)f_1^{(2)}Ad_{F^{(1)}}(b)Ad_{F^{(2)}}(c)S(f_2^{(2)}) \\
&= a \cdot_F (Ad_{F^{(1)}}(b)Ad_{F^{(2)}}(c)) = a \cdot_F (b \cdot_F c),
\end{aligned}$$

and for all  $z \in H_t$ , we have

$$\begin{aligned}
\cdot_F(id \otimes \eta_F)(a \otimes_t z) &= a \cdot_F z = Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(z) \\
&= Ad_{F^{(1)}}(a)\varepsilon_t(F^{(2)}z) = Ad_{zF^{(1)}}(a)\varepsilon_t(F^{(2)}) = Ad_{z1_1}(a)1_2 \\
&= Ad_{z1}(a)z_2 = z_1a\varepsilon_s(z_2) = za = \varepsilon_t(z)a = \varepsilon(1_1z)1_2a \\
&= \varepsilon(1_2z)a1_1 = \varepsilon(1_1z)a1_2 = az = a(Ad_z(1)),
\end{aligned}$$

and

$$\begin{aligned}
\cdot_F(\eta_F \otimes id)(z \otimes_t a) &= z \cdot_F a = Ad_{F^{(1)}}(z)Ad_{F^{(2)}}(a) \\
&= \varepsilon_t(F^{(1)}z)Ad_{F^{(2)}}(a) = \varepsilon_t(F^{(1)})Ad_{F^{(2)}S^{-1}(z)}(a) = S(1_1)Ad_{1_2S^{-1}(z)}(a) \\
&= S(1_1)1_21_1'aS(1_3S^{-1}(z)1_2') = 1_11_21_1'aS(1_2')zS(1_2) = az = a\varepsilon_t(z) \\
&= a\varepsilon(1_1z)1_2 = 1_1a\varepsilon(1_2z) = 1_2a\varepsilon(1_1z) = za = Ad_z(a).
\end{aligned}$$

Finally, we prove that  $\cdot_F$  and  $\eta_F$  are morphisms in  ${}_H\mathcal{M}_F$ . In fact, we have

$$\begin{aligned}
\cdot_F(Ad_h(a \otimes_t b)) &= \cdot_F(Ad_{F^{-(1)}h_1F^{(1)}}(a) \otimes_t Ad_{F^{-(2)}h_2F^{(2)}}(b)) \\
&= Ad_{f^{(1)}F^{-(1)}h_1F^{(1)}}(a)Ad_{f^{(2)}F^{-(2)}h_2F^{(2)}}(b) = Ad_{1_1h_1F^{(1)}}(a)Ad_{1_2h_2F^{(2)}}(b) \\
&= Ad_{h_1F^{(1)}}(a)Ad_{h_2F^{(2)}}(b) = h_1Ad_{F^{(1)}}(a)\varepsilon_s(h_2)Ad_{F^{(2)}}(b)S(h_3) \\
&= h_1\varepsilon_s(h_2)Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(b)S(h_3) = Ad_h(Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(b)) \\
&= Ad_h(a \otimes_F b)
\end{aligned}$$

and

$$\eta_F(Ad_h(x)) = \eta_F(\varepsilon_t(hx)) = \varepsilon_t(hx) = h_1\varepsilon_t(x)S(h_2) = Ad_h(x).$$

This completes the proof.

**Proposition 3.2.** Let  $H$  be a cocommutative weak Hopf algebra,  $F$  a weak invertible unit 2-cocycle. Then  $C_H(H_s)$  is a coalgebra in  ${}_H\mathcal{M}_F$ , which comultiplication and counit are given by

$$\Delta_F : C_H(H_s) \longrightarrow C_H(H_s) \otimes_t C_H(H_s), \quad \Delta_F(a) = Ad_{F^{-}(1)}(a_1) \otimes_t Ad_{F^{-}(2)}(a_2),$$

and

$$\varepsilon_F : C_H(H_s) \longrightarrow H_t, \quad \varepsilon_F(a) = \varepsilon_t(a),$$

for all  $a \in C_H(H_s)$ .

**Proof.** First, we prove that  $\Delta_F$  and  $\varepsilon_F$  are well-defined. Obviously,  $\varepsilon_F$  is well-defined. Now we check as follows, for all  $a \in C_H(H_s)$ , we have

$$Ad_{F^{-}(1)}(a_1) \otimes_t Ad_{F^{-}(2)}(a_2) = Ad_{1_1 F^{-}(1)}(a_1) \otimes Ad_{1_2 F^{-}(2)}(a_2) = Ad_{F^{-}(1)}(a_1) \otimes Ad_{F^{-}(2)}(a_2),$$

Here the second equality uses the cocommutativity.

And, for all  $x \in H_s$ , we have

$$x Ad_{F^{-}(1)}(a_1) = x 1_1 F_1^{-(1)} a_1 S(F_2^{-(1)}) S(1_2) = 1_1 F_1^{-(1)} a_1 S(F_2^{-(1)}) S(1_2) x = Ad_{F^{-}(1)}(a_1) x,$$

so we get  $Ad_{F^{-}(1)}(a_1) \in C_H(H_s)$ . Similarly,  $Ad_{F^{-}(2)}(a_2) \in C_H(H_s)$ . Then  $Ad_{F^{-}(1)}(a_1) \otimes_t Ad_{F^{-}(2)}(a_2) \in C_H(H_s) \otimes_t C_H(H_s)$ .

Secondly, the coassociativity is straightforward, then we verify the counity. In fact, we have

$$\begin{aligned} (\varepsilon_F \otimes id) \Delta_F(a) &= \varepsilon_t(Ad_{F^{-}(1)}(a_1)) \otimes_t Ad_{F^{-}(2)}(a_2) \\ &= \varepsilon_t(Ad_{F^{-}(1)}(a_1)) \otimes_t Ad_{F^{-}(2)}(a_2) = \varepsilon_t(F_1^{-(1)} a_1 S(F_2^{-(1)})) \otimes_t Ad_{F^{-}(2)}(a_2) \\ &= \varepsilon_t(F_1^{-(1)} a_1 \varepsilon_s(F_2^{-(1)})) \otimes_t Ad_{F^{-}(2)}(a_2) = \varepsilon_t(F^{-}(1) 1_1 a_1 S(1_2)) \otimes_t Ad_{F^{-}(2)}(a_2) \\ &= \varepsilon_t(F^{-}(1) a_1 S(1_2)) \otimes_t Ad_{F^{-}(2) S(1_1)}(a_2) = \varepsilon_t(F^{-}(1) a_1 1_1) \otimes_t Ad_{F^{-}(2) 1_2}(a_2) \\ &= \varepsilon_t(F^{-}(1) a_1 S(1_1)) \otimes_t Ad_{F^{-}(2)}(1_2 a_2 S(1_3)) \\ &= \varepsilon_t(F^{-}(1) \varepsilon_t(a S(1_2))_1) \otimes_t Ad_{F^{-}(2)}(1_1 (a S(1_2))_2) \\ &= \varepsilon_t(F^{-}(1) S(1'_1)) \otimes_t Ad_{F^{-}(2)}(1_1 1'_2 a S(1_2)) = \varepsilon_t(F^{-}(1) S(1_1)) \otimes_t Ad_{F^{-}(2) 1_2}(a) \\ &= \varepsilon_t(F^{-}(1) 1_1) \otimes_t Ad_{F^{-}(2) 1_2}(a) = Ad_{\varepsilon_t(F^{-}(1)) F^{-}(2)}(a) \\ &= Ad_1(a) = a \end{aligned}$$

and

$$\begin{aligned} (id \otimes \varepsilon_F) \Delta_F(a) &= Ad_{F^{-}(1)}(a_1) \otimes_t \varepsilon_t(Ad_{F^{-}(2)}(a_2)) \\ &= Ad_{F^{-}(1)}(a_1) \otimes_t \varepsilon_t(F_1^{-(2)} a_2 S(F_2^{-}(2))) = Ad_{F^{-}(1)}(a_1) \otimes_t \varepsilon_t(F_1^{-(2)} a_2 \varepsilon_s(F_2^{-}(2))) \\ &= Ad_{F^{-}(1)}(a_1) \otimes_t \varepsilon_t(F^{-}(2) 1_1 a_2 S(1_2)) = Ad_{F^{-}(1)}(a_1) \otimes_t \varepsilon_t(F^{-}(2) 1_2 a_2 S(1_1)) \end{aligned}$$

$$\begin{aligned}
&= Ad_{F^{-(1)}S^{-1}(1_2)}(a_1) \otimes_t \varepsilon_t(F^{-(2)}a_21_1) = Ad_{F^{-(1)}1_1}(a_1) \otimes_t \varepsilon_t(F^{-(2)}a_2S(1_2)) \\
&= Ad_{F^{-(1)}}(1_1a_1S(1_2)) \otimes_t \varepsilon_t(F^{-(2)}a_2S(1_3)) \\
&= Ad_{F^{-(1)}}(1_1(aS(1_2))_1) \otimes_t \varepsilon_t(F^{-(2)}\varepsilon_t(aS(1_2))_2) \\
&= Ad_{F^{-(1)}}(1_11'_1aS(1_2)) \otimes_t \varepsilon_t(F^{-(2)}1'_2) = Ad_{F^{-(1)}}(1_11'_2aS(1_2)) \otimes_t \varepsilon_t(F^{-(2)}1'_1) \\
&= Ad_{F^{-(1)}}(1_2a) \otimes_t \varepsilon_t(F^{-(2)}1_1) = Ad_{F^{-(1)}1_2}(a) \otimes_t \varepsilon_t(F^{-(2)}1_1) \\
&= Ad_{F^{-(1)}}(a) \otimes_t \varepsilon_t(F^{-(2)}) = Ad_{S^{-1}(\varepsilon_t(F^{-(2)}))F^{-(1)}}(a) = a.
\end{aligned}$$

Finally, we prove that  $\Delta_F$  and  $\varepsilon_F$  are morphisms in  ${}_H\mathcal{M}_F$ . We compute

$$\begin{aligned}
\Delta_F(Ad_h(a)) &= \Delta_F(h_1aS(h_2)) = Ad_{F^{-(1)}}(h_1a_1S(h_4)) \otimes_t Ad_{F^{-(2)}}(h_2a_2S(h_3)) \\
&= Ad_{F^{-(1)}h_1}(a_1) \otimes_t Ad_{F^{-(2)}h_2}(a_2) = Ad_{F^{-(1)}h_11_1}(a_1) \otimes_t Ad_{F^{-(2)}h_21_2}(a_2) \\
&= Ad_{F^{-(1)}h_1F^{(1)}f^{-(1)}}(a_1) \otimes_t Ad_{F^{-(2)}h_2F^{(2)}f^{-(2)}}(a_2) \\
&= Ad_h(Ad_{F^{-(1)}}(a_1) \otimes_t Ad_{F^{-(2)}}(a_2)) = Ad_h(\Delta_F(a))
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_F(Ad_h(a)) &= \varepsilon_t(h_1aS(h_2)) = \varepsilon_t(h_1a\varepsilon_s(h_2)) = \varepsilon_t(h_1\varepsilon_s(h_2)a) = \varepsilon_t(ha) \\
&= h_1\varepsilon_t(a)S(h_2) = Ad_h(\varepsilon_t(a)) = Ad_h(\varepsilon_F(a)),
\end{aligned}$$

as required. This complete the proof of the proposition.

**Theorem 3.3.** Let  $H$  be a cocommutative weak Hopf algebra,  $F$  a weak invertible unit 2-cocycle. Then  $C_H(H_s)$  is a Hopf algebra in  ${}_H\mathcal{M}_F$ , which antipode  $S_F : C_H(H_s) \longrightarrow C_H(H_s)$  is given by  $S_F(a) = S(a)$ , for all  $a \in C_H(H_s)$ .

**Proof.** In order to show that  $C_H(H_s)$  is a Hopf algebra in  ${}_H\mathcal{M}_F$ , we prove that  $\Delta_F$  and  $\varepsilon_F$  are algebra maps in  ${}_H\mathcal{M}_F$  first. It is easy to get the following equality.

$$\begin{aligned}
&f^{(1)}F^{-(1)} \otimes f^{(2)}X^{-(1)}F^{(2)}f^{-(1)} \otimes X^{(1)}X^{-(2)}F^{(1)}F^{-(2)} \otimes X^{(2)}f^{-(2)} \\
&= F_1^{-(1)}F_1^{(1)} \otimes F_2^{-(1)}F_1^{(2)} \otimes F_1^{-(2)}F_2^{(1)} \otimes F_2^{-(2)}F_2^{(2)},
\end{aligned}$$

where  $F = f = X$ , and  $F^{-(1)} = f^{-(1)} = X^{-(1)}$ . For all  $a, b \in C_H(H_s)$ , by the equality, we have

$$\begin{aligned}
\Delta_F(a \otimes_F b) &= \Delta_F(Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(b)) \\
&= Ad_{F^{-(1)}}(F_1^{(1)}a_1S(F_4^{(1)})F_1^{(2)}b_1S(F_4^{(2)})) \otimes_t Ad_{F^{-(2)}}(F_2^{(1)}a_2S(F_3^{(1)})F_2^{(2)}b_2S(F_3^{(2)})) \\
&= Ad_{F^{-(1)}}(Ad_{F_1^{(1)}}(a_1)Ad_{F_1^{(2)}}(b_1)) \otimes_t Ad_{F^{-(2)}}(Ad_{F_2^{(1)}}(a_2)Ad_{F_2^{(2)}}(b_2)) \\
&= F_1^{-(1)}Ad_{F_1^{(1)}}(a_1)Ad_{F_1^{(2)}}(b_1)S(F_2^{-(1)}) \otimes_t Ad_{F^{-(2)}}(Ad_{F_2^{(1)}}(a_2)Ad_{F_2^{(2)}}(b_2)) \\
&= F_1^{-(1)}\varepsilon_s(F_2^{-(1)})Ad_{F_1^{(1)}}(a_1)Ad_{F_1^{(2)}}(b_1)S(F_3^{-(1)}) \otimes_t Ad_{F^{-(2)}}(Ad_{F_2^{(1)}}(a_2)Ad_{F_2^{(2)}}(b_2)) \\
&= F_1^{-(1)}Ad_{F_1^{(1)}}(a_1)\varepsilon_s(F_2^{-(1)})Ad_{F_1^{(2)}}(b_1)S(F_3^{-(1)}) \otimes_t Ad_{F^{-(2)}}(Ad_{F_2^{(1)}}(a_2)Ad_{F_2^{(2)}}(b_2))
\end{aligned}$$

$$\begin{aligned}
&= Ad_{F_1^{-(1)}F_1^{(1)}}(a_1)Ad_{F_2^{-(1)}F_1^{(2)}}(b_1) \otimes_t Ad_{F_1^{-(2)}F_2^{(1)}}(a_2)Ad_{F_2^{-(2)}F_2^{(2)}}(b_2) \\
&= Ad_{f^{(1)}F^{-(1)}}(a_1)Ad_{f^{(2)}X^{-(1)}F^{(2)}f^{-(1)}}(b_1) \otimes_t Ad_{X^{(1)}X^{-(2)}F^{(1)}F^{-(2)}}(a_2)Ad_{X^{(2)}f^{-(2)}}(b_2) \\
&= Ad_{F^{-(1)}}(a_1) \cdot_F Ad_{X^{-(1)}F^{(2)}f^{-(1)}}(b_1) \otimes_t Ad_{X^{-(2)}F^{(1)}F^{-(2)}}(a_2) \cdot_F Ad_{f^{-(2)}}(b_2) \\
&= (Ad_{F^{-(1)}}(a_1) \otimes_t Ad_{F^{-(2)}}(a_2))(Ad_{f^{-(1)}}(b_1) \otimes_t Ad_{f^{-(2)}}(b_2)) \\
&= \Delta_F(a)\Delta_F(b),
\end{aligned}$$

and

$$\begin{aligned}
\Delta_F(1) &= Ad_{F^{-(1)}}(1_1) \otimes_t Ad_{F^{-(2)}}(1_2) = Ad_{F^{-(1)}}(1_1) \otimes_t Ad_{F^{-(2)}1_2}(1) \\
&= Ad_{F^{-(1)}}(1_2) \otimes_t Ad_{F^{-(2)}1_1}(1) = Ad_{F^{-(1)}1_2}(1) \otimes_t Ad_{F^{-(2)}1_1}(1) \\
&= Ad_{F^{-(1)}}(1) \otimes_t Ad_{F^{-(2)}}(1) = \varepsilon_t(F^{-(1)}) \otimes_t \varepsilon_t(F^{-(2)}) \\
&= 1 \otimes_t \varepsilon_t(F^{-(1)})\varepsilon_t(F^{-(2)}) = 1 \otimes_t \varepsilon(1_1F^{-(1)})1_2\varepsilon_t(F^{-(2)}) \\
&= 1 \otimes_t \varepsilon(F^{-(1)})\varepsilon_t(F^{-(2)}) = 1 \otimes_t 1.
\end{aligned}$$

Thus, we obtain  $\Delta_F$  is an algebra map in  ${}_H\mathcal{M}_F$ . Now, we show that  $\varepsilon_F$  is an algebra map as follows:

$$\begin{aligned}
\varepsilon_F(a \cdot_F b) &= \varepsilon_F(Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(b)) = \varepsilon_t(Ad_{F^{(1)}}(a)Ad_{F^{(2)}}(b)) \\
&= \varepsilon_t(Ad_{F^{(1)}}(a)F_1^{(2)}b\varepsilon_s(F_2^{(2)})) = \varepsilon_t(Ad_{F^{(1)}}(a)F^{(2)}1_1bS(1_2)) \\
&= \varepsilon_t(Ad_{F^{(1)}}(a)F^{(2)}b) = \varepsilon(1_1Ad_{F^{(1)}}(a)F^{(2)}b)1_2 \\
&= \varepsilon(\varepsilon_s(F_1^{(1)})aS(F_2^{(1)})1_1F^{(2)}b)1_2 = \varepsilon(1'_1aS(1'_2)S(F^{(1)})1_1F^{(2)}b)1_2 \\
&= \varepsilon(aS(F^{(1)})1_1F^{(2)}b)1_2 = \varepsilon(aS(F^{(1)})1_1)\varepsilon(1_2F^{(2)}b)1_3 \\
&= \varepsilon(aS(F^{(1)})S(1_1))\varepsilon_t(1_2F^{(2)}b) = \varepsilon(aS(F^{(1)}))\varepsilon_t(F^{(2)}b) \\
&= \varepsilon(a\varepsilon_s(F^{(1)}))\varepsilon_t(F^{(2)}b) = \varepsilon(aS(1_2))\varepsilon_t(1_1b) \\
&= v(1_2a)\varepsilon_t(1_1b) = \varepsilon(1_1a)\varepsilon_t(1_2b) = \varepsilon_t(a\varepsilon_t(b)) \\
&= \varepsilon_t(a)\varepsilon_t(b) = \varepsilon_F(a)\varepsilon_F(b).
\end{aligned}$$

So we obtain  $C_H(H_s)$  is a bialgebra in  ${}_H\mathcal{M}_F$ . Then, we want to verify the property of antipode. Obviously,  $S_F$  is well-defined, and  $S_F(Ad_h(a)) = Ad_h(S_F(a))$  holds for all  $h \in H$ . Next, we do the following calculations:

$$\begin{aligned}
\cdot_F(id \otimes S_F)\Delta_F(a) &= Ad_{F^{-(1)}}(a_1) \cdot_F S(Ad_{F^{-(2)}}(a_2)) \\
&= Ad_{F^{(1)}F^{-(1)}}(a_1)Ad_{F^{(2)}}(S(Ad_{F^{-(2)}}(a_2))) \\
&= Ad_{F^{(1)}F^{-(1)}}(a_1)Ad_{F^{(2)}}(S^2(F_2^{-(2)})S(a_2)S(F_1^{-(2)})) \\
&= Ad_{F^{(1)}F^{-(1)}}(a_1)Ad_{F^{(2)}F^{-(2)}}(S(a_2)) = Ad_{1_1}(a_1)Ad_{1_2}(S(a_2)) \\
&= 1_1a_1\varepsilon_s(1_2)S(a_2)S(1_3) = 1_1a_1S(1_3)S(1_2a_2) = 1_2a_1S(1_1)S(1_3a_2) \\
&= (1_2a)_1\varepsilon_t(1_1)S((1_2a)_2) = \varepsilon_t(1_2a1_1) = \varepsilon_t(1_2aS(1_1)) \\
&= \varepsilon_t(1_1aS(1_2)) = \varepsilon_t(a) = \varepsilon_F(a)
\end{aligned}$$



and

$$\begin{aligned}
& \cdot_F(S_F \otimes id)\Delta_F(a) = S(Ad_{F^{-(1)}}(a_1)c_F Ad_{F^{-(2)}}(a_2)) \\
& = Ad_{F^{(1)}}(S(Ad_{F^{-(1)}}(a_1))Ad_{F^{(2)}}(Ad_{F^{-(2)}}(a_2))) \\
& = Ad_{F^{(1)}F^{-(1)}}(S(a_1))Ad_{F^{(2)}F^{-(2)}}(a_2) = 1_1 S(a_1)S(1_2)1_3 a_2 S(1_4) \\
& = 1_1 S(a_1)\varepsilon_s(1_2)S^2(a_2)S(1_3) = 1_1 S(a_2)\varepsilon_s(1_2)S^2(a_1)S(1_3) \\
& = \varepsilon_t(S(a)) = \varepsilon(1_1 S(a))1_2 = \varepsilon(S(a)1_1)1_2 = \varepsilon_t(a) = \varepsilon_F(a).
\end{aligned}$$

This completes our proof.

In this paper, we denote the Hopf algebra we obtained in Theorem 3.3 by  $C_H(H_s)_F$ .

**Corollary 3.4.** Let  $H$  be a cocommutative Hopf algebra,  $F$  a cocycle. Then  $C_H(H_s) = H$ , and  $H_F$  is an  $S$ -Hopf algebra in the sense of [8]. The multiplication and comultiplication are given by

$$h \cdot_F g = Ad_{F^{(1)}}(h)Ad_{F^{(2)}}(g) \quad \text{and} \quad \Delta_F(h) = Ad_{F^{-(1)}}(h_1) \otimes Ad_{F^{-(2)}}(h_2),$$

for all  $h, g \in H_F$ , respectively. The unit, counit and antipode of  $H_F$  coincide with those of  $H$ .

## 4. Isomorphism theorem

Let  $(H, \mathcal{R})$  be a quasitriangular weak Hopf algebra,  $F$  a weak invertible unit 2-cocycle, then by the result of [3] there is a new quasitriangular weak Hopf algebra  $(\tilde{H}, \tilde{\mathcal{R}})$ , defined by the same multiplication, unit and counit, and for all  $h \in \tilde{H}$ ,

$$\tilde{\Delta}(h) = F^{-1}\Delta(h)F, \quad \tilde{\mathcal{R}} = F_{21}^{-1}\mathcal{R}F, \quad \tilde{S}(h) = vS(h)v^{-1},$$

where  $v = F^{(-1)}S(F^{-(2)})$  and  $v^{-1} = S(F^{(1)})F^{(2)}$ .

Applying Corollary 2.6 of the section 2, we obtain that  $C_{\tilde{H}}(\tilde{H}_s)$  is a Hopf algebra in the category  $_{\tilde{H}}\mathcal{M}$  of the left  $\tilde{H}$ -modules, which module action is given by

$$Ad_{\tilde{h}}(g) = h_{\tilde{1}}g\tilde{S}(h_{\tilde{2}}), \quad \text{for all } h \in \tilde{H}, \quad g \in C_{\tilde{H}}(\tilde{H}_s),$$

where  $\tilde{\Delta}(h) = h_{\tilde{1}} \otimes h_{\tilde{2}}$ .

**Lemma 4.1.** The element  $v^{-1} = S(F^{(1)})F^{(2)}$  obeys the following equality:

$$\Delta(v^{-1}) = [(S \otimes S)(F_{21}^{-1})](v^{-1} \otimes v^{-1})F^{-1}.$$

**Proof.** We calculate as follows:

$$\begin{aligned}
\Delta(v^{-1}) &= S(F_2^{(1)})F_1^{(2)} \otimes S(F_1^{(1)})F_2^{(2)} = S(F_2^{(1)})F_1^{(2)}1_1 \otimes S(F_1^{(1)})F_2^{(2)}1_2 \\
&= S(F_2^{(1)})F_1^{(2)}f^{(1)}f^{-(1)} \otimes S(F_1^{(1)})F_2^{(2)}f^{(2)}f^{-(2)} \\
&= S(f_2^{(1)})S(F_2^{(1)})F_3^{(1)}f^{(2)}f^{-(1)} \otimes S(f_1^{(1)})S(F_1^{(1)})F^{(2)}f^{-(2)} \\
&= S(f_2^{(1)})\varepsilon_s(F_2^{(1)})f^{(2)}f^{-(1)} \otimes S(f_1^{(1)})S(F_1^{(1)})F^{(2)}f^{-(2)} \\
&= S(f_2^{(1)})S(1_2)f^{(2)}f^{-(1)} \otimes S(f_1^{(1)})S(F^{(1)}1_1)F^{(2)}f^{-(2)} \\
&= S(1_2f_2^{(1)})f^{(2)}f^{-(1)} \otimes S(1_1f_1^{(1)})S(F^{(1)})F^{(2)}f^{-(2)} \\
&= S(f_2^{(1)})f^{(2)}f^{-(1)} \otimes S(f_1^{(1)})v^{-1}f^{-(2)} \\
&= S(f_2^{(1)}1_2)f^{(2)}f^{-(1)} \otimes S(f_1^{(1)}1_1)v^{-1}f^{-(2)} \\
&= S(f_2^{(1)}F^{(2)}F^{-(2)})f^{(2)}f^{-(1)} \otimes S(f_1^{(1)}F^{(1)}F^{(-1)})v^{-1}f^{-(2)} \\
&= S(F^{-(2)})S(f_1^{(2)}F^{(1)})f_2^{(2)}F^{(2)}f^{-(1)} \otimes S(F^{(-1)})S(f^{(1)})v^{-1}f^{-(2)} \\
&= S(F^{-(2)})S(F^{(1)})\varepsilon_s(f^{(2)})F^{(2)}f^{-(1)} \otimes S(F^{(-1)})S(f^{(1)})v^{-1}f^{-(2)} \\
&= S(F^{-(2)})S(F^{(1)}\varepsilon_s(f^{(2)}))F^{(2)}f^{-(1)} \otimes S(F^{(-1)})S(f^{(1)})v^{-1}f^{-(2)} \\
&= S(\varepsilon_s(f^{(2)})F^{-(2)})S(F^{(1)})F^{(2)}f^{-(1)} \otimes S(F^{(-1)})S(f^{(1)})v^{-1}f^{-(2)} \\
&= S(F^{-(2)})S(F^{(1)})F^{(2)}f^{-(1)} \otimes S(S^{-1}(\varepsilon_s(f^{(2)}))F^{(-1)})S(f^{(1)})v^{-1}f^{-(2)} \\
&= S(F^{-(2)})v^{-1}f^{-(1)} \otimes S(F^{(-1)})\varepsilon_s(f^{(2)})S(f^{(1)})v^{-1}f^{-(2)} \\
&= S(F^{-(2)})v^{-1}f^{-(1)} \otimes S(F^{(-1)})\varepsilon(f^{(2)}1_1)S(1_2)S(f^{(1)})v^{-1}f^{-(2)} \\
&= S(F^{-(2)})v^{-1}f^{-(1)} \otimes S(F^{(-1)})\varepsilon(f^{(2)})S(f^{(1)})v^{-1}f^{-(2)} \\
&= S(F^{-(2)})v^{-1}f^{-(1)} \otimes S(F^{(-1)})v^{-1}f^{-(2)}.
\end{aligned}$$

This finishes our proof.

**Proposition 4.2.** Let  $H$  be cocommutative and  $F$  a weak invertible unit 2-cocycle. Then the category  ${}_{\tilde{H}}\mathcal{M}$  can be identified with the category  ${}_H\mathcal{M}_F$ .

**Proof.** Following [7], it is straightforward.

According to Section 3, we obtain that  $C_H(H_s)_F$  is a Hopf algebra in the category  ${}_H\mathcal{M}_F$ , then we have the following lemma.

**Lemma 4.3.** There is an isomorphism  $\alpha : C_H(H_s)_F \longrightarrow C_{\tilde{H}}(\tilde{H}_s)$  of  $\tilde{H}$ -modules given by  $\alpha(a) = Ad_{F^{(1)}}(a)F^{(2)}$ , for all  $a \in C_H(H_s)_F$ , and  $C_H(H_s)_F \in {}_H\mathcal{M}_F$ ,  $C_{\tilde{H}}(\tilde{H}_s) \in {}_{\tilde{H}}\mathcal{M}$ , where the structures of  $C_H(H_s)_F$  and  $C_{\tilde{H}}(\tilde{H}_s)$  are defined as before.

**Proof.** First of all, we easy to see that  $\alpha(a) \in C_{\tilde{H}}(\tilde{H}_s)$ . In fact, For all  $a \in C_H(H_s)_F$ , we have

$$\begin{aligned}
\tilde{Ad}_1(\alpha(a)) &= 1_{\tilde{1}}\alpha(a)\tilde{S}(1_{\tilde{2}}) = 1_2\alpha(a)\tilde{S}(1_1) = 1_2\alpha(a)vS(1_1)v^{-1} \\
&= 1_2F^{-(1)}aS(F^{-(2)})v^{-1}vS(1_1)v^{-1} = F^{-(1)}aS(F^{-(2)})v^{-1} = \alpha(a).
\end{aligned}$$

Next, we compute another useful form of  $\alpha$  as follows.

$$\begin{aligned}
\alpha(a) &= F_1^{(1)} a S(F_2^{(1)}) F^{(2)} = F_1^{(1)} 1_1 a S(1_2) S(F_2^{(1)}) F^{(2)} \\
&= F_1^{(1)} f^{(1)} f^{-(1)} a S(f^{(2)} f^{-(2)}) S(F_2^{(1)}) F^{(2)} = F_1^{(1)} f^{(1)} f^{-(1)} a S(f^{-(2)}) S(F_2^{(1)} f^{(2)}) F^{(2)} \\
&= F^{(1)} f^{-(1)} a S(f^{-(2)}) S(F_1^{(2)} f^{(1)}) F_2^{(2)} f^{(2)} = F^{(1)} f^{-(1)} a S(f^{-(2)}) S(f^{(1)}) \varepsilon_s(F^{(2)}) f^{(2)} \\
&= 1_2 f^{-(1)} a S(f^{-(2)}) S(f^{(1)}) 1_1 f^{(2)} = 1_2 f^{-(1)} a S(f^{-(2)}) S(f^{(1)} 1_1) f^{(2)} \\
&= 1_2 f^{-(1)} a S(1_1 f^{-(2)}) S(f^{(1)}) f^{(2)} = F^{-(1)} a S(F^{-(2)}) v^{-1}.
\end{aligned}$$

This form of  $\alpha$  imply that  $\alpha$  is invertible with  $\alpha^{-1}(a) = F^{(1)} a v S(F^{(2)})$ . Obviously,  $\alpha^{-1}(a) \in C_H(H_s)_F$ , and we have

$$\begin{aligned}
\alpha \circ \alpha^{-1}(a) &= \alpha(F^{(1)} a v S(F^{(2)})) = F^{-(1)} F^{(1)} a v S(F^{(2)}) S(F^{-(2)}) v^{-1} \\
&= 1_2 a v S(1_1) v^{-1} = 1_2 a F^{-(1)} S(F^{-(2)}) S(1_1) v^{-1} \\
&= 1_1 a S^{-1}(1_2) = a,
\end{aligned}$$

and  $\alpha^{-1} \circ \alpha(a) = a$  is straightforward. The rest proof of the lemma is the same as one in the Hopf case. This finishes the proof.

**Lemma 4.4.** The map  $\alpha : C_H(H_s)_F \longrightarrow C_{\tilde{H}}(\tilde{H}_s)$  in Lemma 4.3 is an algebra map.

**Proof.** For all  $a, b \in C_H(H_s)_F$ , we calculate

$$\begin{aligned}
\alpha(a \cdot_F b) &= Ad_{F^{(1)}}(a) F_1^{(2)} Ad_{f^{(1)}}(b) S(F_2^{(2)}) F_3^{(2)} f^{(2)} \\
&= Ad_{F^{(1)}}(a) F_1^{(2)} Ad_{f^{(1)}}(b) \varepsilon_s(F_2^{(2)}) f^{(2)} \\
&= Ad_{F^{(1)}}(a) F^{(2)} 1_1 Ad_{f^{(1)}}(b) S(1_2) f^{(2)} \\
&= Ad_{F^{(1)}}(a) F^{(2)} Ad_{f^{(1)}}(b) f^{(2)} \\
&= \alpha(a) \alpha(b),
\end{aligned}$$

as required.

**Lemma 4.5.** Let  $H$  be cocommutative and  $F$  a weak invertible unit 2-cocycle. Then the map  $\alpha : C_H(H_s)_F \longrightarrow C_{\tilde{H}}(\tilde{H}_s)$  in Lemma 4.3 is a coalgebra map.

**Proof.** For all  $a \in C_H(H_s)_F$ , firstly, we compute the expression of  $\Delta(\alpha(a))$ .

$$\begin{aligned}
\Delta(\alpha(a)) &= F_1^{-(1)} a_1 S(F_2^{-(1)})(v^{-1})_1 \otimes F_2^{-(1)} a_2 S(F_1^{-(2)})(v^{-1})_2 \\
&= F_1^{-(1)} a_1 S(F_2^{-(1)}) S(f^{-(2)}) v^{-1} X^{-(1)} \otimes F_2^{-(1)} a_2 S(F_1^{-(2)}) S(f^{-(1)}) v^{-1} X^{-(2)} \\
&= [F_1^{-(1)} f_1^{-(1)} a_1 S(f^{-(2)}) \otimes F_2^{-(1)} f_2^{-(1)} a_2 S(F^{-(2)} f_3^{-(1)})](v^{-1} \otimes v^{-1}) F^{-1}.
\end{aligned}$$

Then, we have

$$\tilde{\Delta}(\alpha(a)) = F^{-(1)} \alpha(a)_1 F^{(1)} \otimes F^{-(2)} \alpha(a)_2 F^{(2)}.$$

Next, applying Corollary 2.6 to  $(\tilde{H}, \tilde{\mathcal{R}})$ , we obtain the form of the comultiplication of  $C_{\tilde{H}}(\tilde{H}_s)$ , then

$$\begin{aligned}
\widehat{\tilde{\Delta}}(\alpha(a)) &= F^{-(1)}\alpha(a)_1 F^{(1)}\tilde{R}^{(2)}\tilde{S}(\tilde{r}^{(2)}) \otimes_t \tilde{r}^{(1)} F^{-(2)}\alpha(a)_2 F^{(2)}\tilde{R}^{(1)} \\
&= F^{-(1)}\alpha(a)_1 F^{(1)}\tilde{R}^{(2)}vS(\tilde{r}^{(2)})v^{-1} \otimes_t \tilde{r}^{(1)} F^{-(2)}\alpha(a)_2 F^{(2)}\tilde{R}^{(1)} \\
&= F^{-(1)}\alpha(a)_1 F^{(1)}f^{-(1)}f^{(2)}vS(X^{-(1)}X^{(2)})v^{-1} \otimes_t X^{-(2)}X^{(1)}F^{-(2)} \\
&\quad \alpha(a)_2 F^{(2)}f^{-(2)}f^{(1)} \\
&= F^{-(1)}\alpha(a)_1 1_1 f^{(2)}vS(F^{(2)})S(f^{-(1)})v^{-1} \otimes_t f^{-(2)}F^{(1)}F^{-(2)}\alpha(a)_2 1_2 f^{(1)} \\
&= F^{-(1)}\alpha(a)_2 f^{(2)}vS(F^{(2)})S(f^{-(1)})v^{-1} \otimes_t f^{-(2)}F^{(1)}F^{-(2)}\alpha(a)_1 f^{(1)} \\
&= F^{-(1)}X_2^{-(1)}F_2'^{-(1)}a_2S(F_3'^{-(1)})S(X^{-(2)})v^{-1}f'^{(2)}f^{(2)}vS(F^{(2)})S(f^{-(1)})v^{-1} \\
&\quad \otimes_t f^{-(2)}F^{(1)}F^{-(2)}X_1^{-(1)}F_1'^{-(1)}a_1S(F'^{-(2)})v^{-1}f^{(1)}f^{(1)} \\
&= F^{-(1)}X_2^{-(1)}Ad_{F_2'^{-(1)}}(a_2)S(X^{-(2)})v^{-1}1_1vS(F^{(2)})S(f^{-(1)})v^{-1} \\
&\quad \otimes_t f^{-(2)}F^{(1)}F^{-(2)}X_1^{-(1)}F_1'^{-(1)}a_1S(F'^{-(2)})v^{-1}1_2.
\end{aligned}$$

For convenience, we verify the following equations. For all  $y \in H_s$ ,  $z \in H_t$ ,

$$\begin{aligned}
v^{-1}yv &= v^{-1}yF^{-(1)}S(F^{-(2)}) = v^{-1}F^{-(1)}S(F^{-(2)}y) = v^{-1}F^{-(1)}S(y)S(F^{-(2)}) \\
&= v^{-1}F^{-(1)}S(S(y)F^{-(2)}) = v^{-1}vS^2(y) = S^2(y), \\
v^{-1}z &= S(F^{(1)})F^{(2)}z = S(zF^{(1)})F^{(2)} = S(F^{(1)})S(z)F^{(2)} = S(F^{(1)}S(z))F^{(2)} \\
&= S^2(z)S(F^{(1)})F^{(2)} = S^2(z)v^{-1}.
\end{aligned}$$

i.e.,  $v^{-1}yv = S^2(y)$  and  $v^{-1}z = S^2(z)v^{-1}$ .

We now proceed to compute  $\widehat{\tilde{\Delta}}(\alpha(a))$ :

$$\begin{aligned}
\widehat{\tilde{\Delta}}(\alpha(a)) &= F^{-(1)}X_2^{-(1)}Ad_{F_2'^{-(1)}}(a_2)S(X^{-(2)})S(1_2)S(F^{(2)})S(f^{-(1)})v^{-1} \\
&\quad \otimes_t f^{-(2)}F^{(1)}F^{-(2)}X_1^{-(1)}F_1'^{-(1)}a_1S(F'^{-(2)})S(1_1)v^{-1} \\
&= F^{-(1)}X_1^{-(1)}Ad_{F_1'^{-(1)}}(a_1)S(F^{(2)}1_2X^{-(2)})S(f^{-(1)})v^{-1} \\
&\quad \otimes_t f^{-(2)}F^{(1)}F^{-(2)}X_2^{-(1)}F_2'^{-(1)}a_2S(1_1F'^{-(2)})v^{-1} \\
&= X^{-(1)}Ad_{F_1'^{-(1)}}(a_1)S(X_2^{-(2)})S(F^{-(2)})S(F^{(2)}1_2)S(f^{-(1)})v^{-1} \\
&\quad \otimes_t f^{-(2)}F^{(1)}F^{-(1)}X_1^{-(2)}F_2'^{-(1)}a_2S(1_1F'^{-(2)})v^{-1} \\
&= X^{-(1)}Ad_{F_1'^{-(1)}}(a_1)S(X_1^{-(2)})S(F^{-(2)})S(F^{(2)})S(f^{-(1)})v^{-1} \\
&\quad \otimes_t f^{-(2)}1_2F^{(1)}F^{-(1)}X_2^{-(2)}F_2'^{-(1)}a_2S(1_1F'^{-(2)})v^{-1} \\
&= F^{-(1)}Ad_{F_1'^{-(1)}}(a_1)S(F_1^{-(2)})S(1_2)S(f^{-(1)})v^{-1} \\
&\quad \otimes_t f^{-(2)}1_2'1_1F_2^{-(2)}F_2'^{-(1)}a_2S(1_1'F'^{-(2)})v^{-1} \\
&= F^{-(1)}Ad_{F_1'^{-(1)}}(a_1)S(f^{-(1)}(1_2F^{-(2)})_1)v^{-1} \otimes_t f^{-(2)}(1_2F^{-(2)})_2
\end{aligned}$$

$$\begin{aligned}
& F_2'^{(1)} a_2 S(1_1 F'^{(2)}) v^{-1} \\
= & F^{-(1)} 1_2 \text{Ad}_{F_1'^{(1)}}(a_1) S(f^{-(1)} F_1^{-(2)}) v^{-1} \otimes_t f^{-(2)} F_2^{-(2)} \\
& F_2'^{(1)} a_2 S(1_1 F'^{(2)}) v^{-1} \\
= & f^{-(1)} F_1^{-(1)} 1_2 \text{Ad}_{F_1'^{(1)}}(a_1) S(f^{-(2)} F_2^{-(1)}) v^{-1} \otimes_t F^{-{(2)}} \\
& F_2'^{(1)} a_2 S(1_1 F'^{(2)}) v^{-1} \\
= & f^{-(1)} F_1^{-(1)} 1_2 \text{Ad}_{F_1'^{(1)}}(a_1) S(1_3) S(F_2^{-(1)}) S(f^{-(2)}) v^{-1} \\
& \otimes_t F^{-{(2)}} F_2'^{(1)} a_2 S(1_1 F'^{(2)}) v^{-1} \\
= & f^{-(1)} \text{Ad}_{F^{-{(1)}} 1_2 F_1'^{(1)}}(a_1) S(f^{-(2)}) v^{-1} \otimes_t F^{-{(2)}} F_2'^{(1)} \\
& a_2 S(1_1 F'^{(2)}) v^{-1} \\
= & f^{-(1)} \text{Ad}_{F^{-{(1)}} F_1'^{(1)}}(a_1) S(f^{-(2)}) v^{-1} \otimes_t 1_2 F^{-{(2)}} F_2'^{(1)} \\
& a_2 S(1_1 F'^{(2)}) v^{-1} \\
= & f^{-(1)} \text{Ad}_{F'^{(1)}}(a_1) S(f^{-(2)}) v^{-1} \otimes_t 1_2 F^{-{(1)}} F_1'^{(2)} a_2 \\
& S(1_1 F^{(2)} F_2'^{(2)}) v^{-1} \\
= & f^{-(1)} \text{Ad}_{F'^{(1)}}(a_1) S(f^{-(2)}) v^{-1} \otimes_t 1_2 F^{-{(1)}} \text{Ad}_{F'^{(2)}}(a_2) \\
& S(1_1 F^{(2)}) v^{-1} \\
= & f^{-(1)} \text{Ad}_{F'^{(1)}}(a_1) S(f^{-(2)}) v^{-1} \otimes_t F^{-{(1)}} \text{Ad}_{F'^{(2)}}(a_2) S(F^{(2)}) v^{-1} \\
= & (\alpha \otimes \alpha) \Delta_F(a).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\widehat{\varepsilon} \circ \alpha(a) &= \varepsilon_t(\alpha(a)) = \varepsilon_t(F^{-{(1)}} a S(F^{-{(2)}}) v^{-1}) \\
&= \varepsilon_t(F^{-{(1)}} a S(F^{-{(2)}}) S(F^{(1)}) F^{(2)}) \\
&= \varepsilon_t(F^{-{(1)}} F_1^{(1)} a S(F_2^{(1)}) S(F_1^{-(2)}) F_2^{-(2)} F^{(2)}) \\
&= \varepsilon_t(F^{-{(1)}} \text{Ad}_{F^{(1)}}(a) \varepsilon_s(F^{-{(2)}}) F^{(2)}) \\
&= \varepsilon_t(1_1 \text{Ad}_{F^{(1)}}(a) S(1_2) F^{(2)}) = \varepsilon_t(\text{Ad}_{F^{(1)}}(a) F^{(2)}) \\
&= \varepsilon_t(\text{Ad}_{F^{(1)}}(a) \varepsilon_t(F^{(2)})) = \varepsilon_t(\text{Ad}_{1_1}(a) 1_2) \\
&= \varepsilon_t(1_1 a \varepsilon_s(1_2)) = \varepsilon_t(a) = \varepsilon_F(a).
\end{aligned}$$

This completes the proof.

**Lemma 4.6.** Let  $H$  be cocommutative and  $F$  a weak invertible unit 2-cocycle. Then the antipode  $S_F$  on  $C_H(H_s)_F$  and the antipode  $\widehat{\widetilde{S}}$  on  $C_{\widehat{\widetilde{H}}}(\widetilde{H}_s)$  satisfy the following condition:

$$\widehat{\widetilde{S}} \circ \alpha = \alpha \circ S_F.$$

**Proof.** In order to verification, we calculate as follows:

$$\alpha^{-1} \circ \widehat{\widetilde{S}} \circ \alpha(a)$$

$$\begin{aligned}
&= F^{(1)} f^{-(1)} f^{(2)} X^{- (1)} S(X^{- (2)}) S(X^{(2)}) Ad_{X^{(1)} f^{-(2)} f^{(1)}}(S(a)) S(F^{(2)}) \\
&= F^{(1)} f^{-(1)} X_2^{- (1)} f_1^{(2)} S(X^{- (2)} f_2^{(2)}) S(X^{(2)}) Ad_{X^{(1)} f^{-(2)} X_1^{- (1)} f^{(1)}}(S(a)) S(F^{(2)}) \\
&= F^{(1)} f^{-(1)} X_2^{- (1)} \varepsilon_t(f^{(2)}) S(X^{- (2)}) S(X^{(2)}) Ad_{X^{(1)} f^{-(2)} X_1^{- (1)} f^{(1)}}(S(a)) S(F^{(2)}) \\
&= F^{(1)} f^{-(1)} X_2^{- (1)} 1_2 S(X^{- (2)}) S(X^{(2)}) Ad_{X^{(1)} f^{-(2)} X_1^{- (1)} 1_1}(S(a)) S(F^{(2)}) \\
&= F^{(1)} f^{-(1)} X_1^{- (1)} S(X^{(2)} X^{- (2)}) Ad_{X^{(1)} f^{-(2)} X_2^{- (1)}}(S(a)) S(F^{(2)}) \\
&= F^{(1)} X^{- (1)} S(X^{(2)} f^{- (2)} X_2^{- (2)}) Ad_{X^{(1)} f^{- (1)} X_1^{- (2)}}(S(a)) S(F^{(2)}) \\
&= F^{(1)} X^{- (1)} S(1_2 X_2^{- (2)}) Ad_{1_1 X_1^{- (2)}}(S(a)) S(F^{(2)}) \\
&= F^{(1)} f^{-(1)} S(f_2^{- (2)}) Ad_{f_1^{- (2)}}(S(a)) S(F^{(2)}) \\
&= F^{(1)} f^{-(1)} S(f_3^{- (2)}) f_1^{- (2)} S(a) S(f_2^{- (2)}) S(F^{(2)}) \\
&= F^{(1)} f^{-(1)} S(f_1^{- (2)}) f_2^{- (2)} S(a) S(f_3^{- (2)}) S(F^{(2)}) \\
&= F^{(1)} f^{-(1)} \varepsilon_s(f_1^{- (2)}) S(a) S(f_2^{- (2)}) S(F^{(2)}) \\
&= F^{(1)} f^{-(1)} 1_1 S(a) S(1_2) S(f^{- (2)}) S(F^{(2)}) \\
&= 1_1 S(a) S(1_2) = S(a) = S_F(a).
\end{aligned}$$

Thus we have  $\widehat{\widetilde{S}} \circ \alpha = \alpha \circ S_F$ . This finishes the proof.

The following isomorphism theorem is the main result of this paper.

**Theorem 4.7.** Let  $H$  be cocommutative and  $F$  a weak invertible unit 2-cocycle. Assume that  $C_H(H_s)_F$  be the Hopf algebra with the structure as shown in Theorem 3.3 viewed as an object in the category  $_{\widetilde{H}}\mathcal{M}$  of  $\widetilde{H}$ -modules. In this category, there is an isomorphism of Hopf algebras

$$\alpha : C_H(H_s)_F \cong C_{\widetilde{H}}(\widetilde{H}_s).$$

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